

Non-homogeneous systems of hydrodynamic type possessing Lax representations

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Abstract. We consider $1 + 1$ - dimensional non-homogeneous systems of hydrodynamic type that possess Lax representations with movable singularities. We present a construction, which provides a wide class of examples of such systems with arbitrary number of components. In the two-component case a classification is given.

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1 Introduction

The integrability theory for $1+1$ - dimensional homogeneous systems of hydrodynamic type of the form

$$u_{i,t} = \sum_{j=1}^n a_{ij}(\mathbf{u}) u_{j,x}, \quad i = 1, \dots, n, \quad (1.1)$$

where $\mathbf{u} = (u_1, \dots, u_n)$ was developed in [1, 2]. In the case of $1+2$ - dimensional homogeneous systems a definition of integrability based on the existence of hydrodynamic reductions was proposed in [3]. For examples and a classification of integrable cases see [4] and references therein.

In contrast to the homogeneous case, there is no satisfactory criteria of integrability for non-homogeneous hydrodynamic type systems of the form

$$u_{i,t} = \sum_{j=1}^n a_{ij}(\mathbf{u}) u_{j,x} + b_i(\mathbf{u}), \quad i = 1, \dots, n. \quad (1.2)$$

On the other hand, the following example

$$u_t = vu_x + \frac{1}{v-u}, \quad v_t = uv_x + \frac{1}{u-v} \quad (1.3)$$

found in [6] indicates the existence of integrable systems (1.2) having properties unusual for $1+1$ -dimensional integrable models. In particular, system (1.3) has only a few local infinitesimal symmetries and therefore the symmetry approach to classification of integrable $1+1$ -dimensional systems [7] is not applicable for systems similar to (1.3). It turns out that this system possesses infinite non-commutative hierarchy of nonlocal symmetries explicitly depending on x and t . These symmetries look like symmetries of the so called Gibbons-Tsarev systems found in [5].

Notice that system (1.3) was also derived in another context, in [8, 9] where conditions for two quadratic Hamiltonians to be in involution were studied.

It was shown in [10] that the general solution of (1.3) can be described in terms of a conformal mapping of a slit domain to the upper half plane (cf. [11, 12]). A class of special solutions was constructed in [13]. All these results are related to the existence of the following Lax representation

$$\Psi_x = \frac{\lambda - u - v}{(\lambda - u)(\lambda - v)} \Psi_\lambda, \quad \Psi_t = \frac{1}{(\lambda - u)(\lambda - v)} \Psi_\lambda \quad (1.4)$$

for (1.3). Here λ is a spectral parameter.

In this paper we consider systems (1.2) having Lax representations of the form

$$\Psi_x = f(\mathbf{u}, \lambda) \Psi_\lambda, \quad \Psi_t = g(\mathbf{u}, \lambda) \Psi_\lambda. \quad (1.5)$$

For brevity, we call such systems *integrable*. Lax pairs similar to (1.5) were considered in [14]. Note that Lax representations with λ -derivatives of the Ψ -function are typical for Painleve type equations. Possibly the technique developed in [16] for solving of dispersionless systems can be adopted to systems with such Lax pairs.

To generalize example (1.3), (1.4) we assume (cf. [15]) that both functions f and g in (1.5) have simple poles at $\lambda = u_1, \dots, u_n$. In general, we do not restrict ourselves to functions f, g rational in λ but all examples in this paper are rational.

In Section 2 we show that for any system (1.2) with a Lax representation of the kind described above the matrix $\mathbf{a} = (a_{ij}(\mathbf{u}))$ is diagonal and weakly non-linear [17]. The latter property means that the functions $a_{ii}(\mathbf{u})$ do not depend on u_i . We establish a partial separation of variables for functions f and g in (1.5) and write down a general functional equation describing these functions.

In Section 3 we construct a large class of examples of integrable systems of the form (1.2) with arbitrary n . These systems depend on parameters. The main continuous parameters are the roots λ_i of a polynomial with constant coefficients. To each λ_i a non-negative integer k_i is attached. In Section 3.1 the case is considered when all the k_i are zero. In Section 3.2 we describe a limit of formulas from Section 3.1 when some of the λ_i coincide. In Section 3.3 we generalize results obtained in Section 3.1 to the case of arbitrary k_i .

Section 4 is devoted to the case $n = 2$. We find all possible pairs of functions f and g appearing in the Lax representation (1.5). Even for $n = 2$ the functional equation for functions f and g is highly non-trivial. It turns out that it is almost equivalent to a functional equation describing the functions F, H in the fieldless Gibbons-Tsarev type systems

$$\partial_i \xi_j = F(\xi_i, \xi_j) \partial_i u, \quad \partial_i \partial_j u = H(\xi_i, \xi_j) \partial_i u \partial_j u, \quad i \neq j, \quad i, j = 1, \dots, N. \quad (1.6)$$

Here u, ξ_i are functions of variables r^1, \dots, r^N , and $\partial_i = \frac{\partial}{\partial r^i}$. Such systems play a crucial role [3, 4] in the integrability theory for 1+2-dimensional homogeneous hydrodynamic type systems. As an additional result we find new examples of Gibbons-Tsarev type systems (1.6). From our classification it follows that in the case $n = 2$ all systems with Lax representations (1.5) can be constructed by the approach developed in Section 3.

In Section 5 we present some explicit examples of integrable systems of the form (1.2) in the case $n = 3$.

2 Lax pairs with movable singularities

Our goal is a generalization of example (1.3), (1.4). In this example both functions f and g in (1.5) have simple poles at $\lambda = u$ and $\lambda = v$. In the general case let us assume (cf. [15]) that the functions f and g in (1.5) have simple poles at $\lambda = u_1, \dots, u_n$.

Computing $(\Psi_x)_t$ and $(\Psi_t)_x$ by virtue of (1.2) and (1.5) and equating coefficients at $u_{j,x}$, $j =$

$1, \dots, n$, we obtain the following compatibility conditions:

$$\frac{\partial g}{\partial u_i} = \sum_{j=1}^n a_{ji} \frac{\partial f}{\partial u_j}, \quad i = 1, \dots, n, \quad (2.7)$$

$$\sum_{i=1}^n b_i \frac{\partial f}{\partial u_i} + f \frac{\partial g}{\partial \lambda} - g \frac{\partial f}{\partial \lambda} = 0. \quad (2.8)$$

Computing the singular part of (2.7) at $\lambda = u_j$, where $j \neq i$, we obtain $a_{ji} = \delta_{i,j} a_i$. We assume that the functions a_i are pairwise distinct. Now our system (1.2) reads:

$$u_{i,t} = a_i(\mathbf{u}) u_{i,x} + b_i(\mathbf{u}), \quad i = 1, \dots, n. \quad (2.9)$$

The compatibility condition (2.7) becomes

$$\frac{\partial g}{\partial u_i} = a_i \frac{\partial f}{\partial u_i}, \quad i = 1, \dots, n. \quad (2.10)$$

Let

$$f = \frac{\phi_i}{\lambda - u_i} + O(\lambda - u_i), \quad g = \frac{\psi_i}{\lambda - u_i} + O(\lambda - u_i), \quad \phi_i \neq 0, \quad \psi_i \neq 0$$

for $\lambda \rightarrow u_i$. Computing the singular parts of (2.10), we obtain

$$\phi_i = a_i \psi_i, \quad \frac{\partial \phi_j}{\partial u_i} = a_i \frac{\partial \psi_j}{\partial u_i}$$

for all $i, j = 1, \dots, n$. Substituting the first of these equations into the second one, we obtain $(a_j \psi_j)_{u_i} = a_i \psi_{j,u_i}$. From here it follows that if $i = j$, then

$$\frac{\partial a_i}{\partial u_i} = 0,$$

otherwise

$$\frac{\partial \psi_j}{\partial u_i} = \frac{1}{a_i - a_j} \frac{\partial a_j}{\partial u_i} \psi_j, \quad i \neq j.$$

The first of these equations shows that the homogeneous system

$$u_{i,t} = a_i u_{i,x}, \quad i = 1, \dots, n \quad (2.11)$$

is weakly non-linear. The compatibility conditions for the second equation means that this homogeneous system is semi-Hamiltonian. Weakly non-linear semi-Hamiltonian systems (2.11) were studied by E. Ferapontov in [17]. In particular, he classified these systems finding possible coefficients a_i in a closed form in terms of arbitrary functions of one variable. Namely, for any weakly non-linear semi-Hamiltonian system (2.11) the coefficient a_i has the form:

$$a_i = \frac{\det \Delta_{n,i}}{\det \Delta_{n-1,i}}. \quad (2.12)$$

Here and in the sequel

$$\Delta = \begin{pmatrix} 1 & \dots & 1 \\ q_{1,1}(u_1) & \dots & q_{1,n}(u_n) \\ \dots & \dots & \dots \\ q_{n-1,1}(u_1) & \dots & q_{n-1,n}(u_n) \end{pmatrix}, \quad (2.13)$$

where $q_{i,j}$ are arbitrary functions of one variable. The meaning of the indexes on Δ in (2.12) is as follows. For any matrix M we denote by $M_{i,j}$ its i, j -minor. In other words, $M_{i,j}$ is obtained from M by deleting its i -th row and j -th column. By M_i we denote the matrix obtained from M by deleting its i -th row.

For functions (2.12) the general solution of (2.10) is given by

$$f = \frac{\det P}{\det \Delta}, \quad g = \frac{\det Q}{\det \Delta}, \quad (2.14)$$

where

$$P = \begin{pmatrix} h_1(\lambda, u_1) & \dots & h_n(\lambda, u_n) \\ \Delta_{n-1} \end{pmatrix}, \quad Q = \begin{pmatrix} h_1(\lambda, u_1) & \dots & h_n(\lambda, u_n) \\ \Delta_n \end{pmatrix}.$$

Here h_1, \dots, h_n are arbitrary functions of two variables. Notice that $P_n = Q_n$. According to our assumption each function $h_i(\lambda, u)$ has a simple pole at $\lambda = u$. Substituting these expressions for f, g into (2.8) and computing the singular part at $\lambda = u_i$, we get

$$b_i = \frac{\det Q_{n,i}|_{\lambda=u_i}}{\det \Delta_{n-1,i}}, \quad i = 1, \dots, n. \quad (2.15)$$

Substituting these into (2.8), we obtain a functional equation, which can be written in the following form:

$$\det \begin{pmatrix} h_1(\lambda, u_1)_\lambda & \dots & h_n(\lambda, u_n)_\lambda \\ Q_n \end{pmatrix} + \sum_{i=1}^n (-1)^{i-1} h_i(\lambda, u_i)_{u_i} \det Q_{n,i}|_{\lambda=u_i} + \quad (2.16)$$

$$\frac{1}{\det \Delta} \sum_{1 \leq k_1 \leq n-1, 1 \leq k_2 \leq n} (-1)^{k_1+k_2} \det Q_{n,k_2}|_{\lambda=u_{k_2}} \det \begin{pmatrix} h_1(\lambda, u_1) & \dots & h_n(\lambda, u_n) \\ \Delta_{k_1+1} \end{pmatrix} q'_{k_1,k_2}(u_{k_2}) = 0.$$

Remark 1. Expanding numerators of (2.14) by the first row, we get

$$f = h_1(\lambda, u_1)\phi_1 + \dots + h_n(\lambda, u_n)\phi_n, \quad g = h_1(\lambda, u_1)\psi_1 + \dots + h_n(\lambda, u_n)\psi_n,$$

where

$$\phi_i = (-1)^{i-1} \frac{\Delta_{n-1,i}}{\Delta_{n-1}}, \quad \psi_i = (-1)^{i-1} \frac{\Delta_{n,i}}{\Delta_n}.$$

Note that according to the Cramer rule, ϕ_i, ψ_i are the solutions of the following system of linear equations:

$$q_{j,1}\phi_1 + \dots + q_{j,n}\phi_n = \delta_{n-2,j}, \quad q_{j,1}\psi_1 + \dots + q_{j,n}\psi_n = \delta_{n-1,j}, \quad (2.17)$$

where $j = 0, \dots, n-1$, $q_{0,i}(u) = 1$ and $\delta_{i,j}$ is the Kronecker delta. \square

Remark 2. For all known examples the functions $h_i(\lambda, u)$, $q_{j,i}(u)$ don't depend on i . We conjecture that this property holds for any solution of (2.16). This conjecture is proved in the cases $n = 2, 3$. \square

3 A class of integrable systems with arbitrary number of components

The compatibility condition for (1.5) reads¹

$$\frac{\partial f}{\partial t} + f \frac{\partial g}{\partial \lambda} = \frac{\partial g}{\partial x} + g \frac{\partial f}{\partial \lambda}. \quad (3.18)$$

In this section we assume that f and g are rational functions in λ . In addition, we suppose that both f and g are divisible by a polynomial $S(\lambda)$ with constant coefficients. At first glance this is not true for our basic example (1.4). Nevertheless, under the transformation

$$\lambda \rightarrow \bar{\lambda} = \frac{a\lambda + b}{c\lambda + d} \quad (3.19)$$

with constant a, b, c, d a common multiplier $(c\lambda + d)^3$ in f and g appears.

One can construct conservation laws for a system of the form (1.2) using such a Lax representation (1.5) as follows:

Lemma 1. Suppose that $f = S(\lambda)f_1$, $g = S(\lambda)g_1$, where $S(\lambda)$ is a polynomial with constant coefficients and with roots $\lambda_1, \dots, \lambda_p$. Set $\mu_i = f_1|_{\lambda=\lambda_i}$, and $\nu_i = g_1|_{\lambda=\lambda_i}$. Then

$$\frac{\partial \mu_i}{\partial t} = \frac{\partial \nu_i}{\partial x}. \quad (3.20)$$

Moreover, if λ_i is a root of multiplicity k and $\mu_{i,j} = \frac{\partial^j f_1}{\partial \lambda^j}|_{\lambda=\lambda_i}$, $\nu_{i,j} = \frac{\partial^j g_1}{\partial \lambda^j}|_{\lambda=\lambda_i}$, where $j = 0, \dots, k-1$, then

$$\frac{\partial \mu_{i,j}}{\partial t} = \frac{\partial \nu_{i,j}}{\partial x}. \quad (3.21)$$

Proof. Substituting our expressions for f , g into (3.18), we get

$$\frac{\partial f_1}{\partial t} + S(\lambda) \frac{\partial g_1}{\partial \lambda} f_1 = \frac{\partial g_1}{\partial x} + S(\lambda) \frac{\partial f_1}{\partial \lambda} g_1. \quad (3.22)$$

To complete the proof we set $\lambda = \lambda_i$ here. In the case of multiple roots we compute derivatives of (3.22) with respect to λ and then set $\lambda = \lambda_i$. \square

¹Note that this equation can be written as $\frac{\partial f}{\partial t} - \frac{\partial g}{\partial x} + [f, g] = 0$, where $[f, g] = f \frac{\partial g}{\partial \lambda} - g \frac{\partial f}{\partial \lambda}$ is the bracket in an algebra of vector fields. Therefore (3.18) can be considered as zero curvature equation for a connection in an $Diff^1$ -bundle, where $Diff^1$ is the group of diffeomorphisms of one-dimensional manifold with coordinate λ .

3.1 Principal series of examples

Let

$$f = \frac{S(\lambda)P(\lambda)}{R(\lambda)}, \quad g = \frac{S(\lambda)Q(\lambda)}{R(\lambda)}, \quad (3.23)$$

where S , P , Q , R are polynomials in λ and all coefficients of S are constant. Then equation (3.18) reads as

$$R \frac{\partial P}{\partial t} - P \frac{\partial R}{\partial t} + Q \frac{\partial R}{\partial x} - R \frac{\partial Q}{\partial x} + S(\lambda) \left(P \frac{\partial Q}{\partial \lambda} - Q \frac{\partial P}{\partial \lambda} \right) = 0. \quad (3.24)$$

Proposition 1. Let

$$h_i(\lambda, u) = \frac{S(\lambda)}{\lambda - u}, \quad q_{j,i}(u) = u^j,$$

if $i = 1, \dots, n$, $j = 1, \dots, n - m - 3$ and

$$q_{n-m-3+j,i}(u) = \sum_{k=1}^{n+1-m} \frac{c_{k,j}}{u - \lambda_k} \quad (3.25)$$

if $i = 1, \dots, n$, $j = 1, \dots, m + 2$. Here m is a fixed integer such that $0 \leq m \leq \frac{n-1}{2}$, $S(\lambda) = (\lambda - \lambda_1) \dots (\lambda - \lambda_{n+1-m})$, where λ_i are pairwise distinct constants, and $c_{i,j}$ are arbitrary constants such that the matrix $(c_{i,j})$ has rank $m + 2$. Then formulas (2.13) - (2.15) define a system (2.9) possessing a Lax representation (1.5).

Proof. Expanding the determinants in the numerators of (2.14) by the first row, we get

$$f = S(\lambda) \left(\frac{\phi_1}{\lambda - u_1} + \dots + \frac{\phi_n}{\lambda - u_n} \right), \quad g = S(\lambda) \left(\frac{\psi_1}{\lambda - u_1} + \dots + \frac{\psi_n}{\lambda - u_n} \right), \quad (3.26)$$

where $\phi_1, \dots, \phi_n, \psi_1, \dots, \psi_n$ satisfy the linear system (2.17) (see Remark 1 in Section 2). The part of this system corresponding to $j = 0, \dots, n - m - 3$ has the form

$$u_1^j \phi_1 + \dots + u_n^j \phi_n = 0, \quad u_1^j \psi_1 + \dots + u_n^j \psi_n = 0. \quad (3.27)$$

From these equations it follows that

$$f = \frac{S(\lambda)P(\lambda)}{(\lambda - u_1) \dots (\lambda - u_n)}, \quad g = \frac{S(\lambda)Q(\lambda)}{(\lambda - u_1) \dots (\lambda - u_n)}, \quad (3.28)$$

where P , Q are polynomials in λ of degree $m+1$. Indeed, using expressions (3.26) and expanding $\frac{f}{S}$ and $\frac{g}{S}$ with respect to powers of λ^{-1} , we get left hand sides of (3.27) as coefficients at λ^{-j} , $j = 1, \dots, n - m - 2$. The left hand side of (3.24) is a polynomial in λ of degree $n + m + 1$. Equating its coefficients to zero, we obtain $n + m + 2$ non-homogeneous differential equations

of hydrodynamic type for u_1, \dots, u_n . To obtain a system of the form (1.2) we have to prove that there exist $m + 2$ linear dependence relations between these equations. Let

$$\mu_i = \frac{\phi_1}{\lambda_i - u_1} + \dots + \frac{\phi_n}{\lambda_i - u_n}, \quad \nu_i = \frac{\psi_1}{\lambda_i - u_1} + \dots + \frac{\psi_n}{\lambda_i - u_n}, \quad (3.29)$$

where $i = 1, \dots, n + 1 - m$. According to Lemma 1, we have $n + 1 - m$ differential equations (3.20). On the other hand, the relations

$$c_{1,j}\mu_1 + \dots + c_{n+1-m,j}\mu_{n+1-m} = \delta_{m+1,j}, \quad c_{1,j}\nu_1 + \dots + c_{n+1-m,j}\nu_{n+1-m} = \delta_{m+2,j} \quad (3.30)$$

for $j = 1, \dots, m + 2$, are fulfilled. Indeed, substituting the expressions (3.29) for μ_i, ν_i into (3.30), we get the part of the linear system (2.17) corresponding to $j = n - m - 2, \dots, n - 1$, where the functions $q_{j,i}$ are given by (3.25). Relations (3.30) give us $m + 2$ linear dependence relations between equations (3.20), namely

$$c_{1,j}((\mu_1)_t - (\nu_1)_x) + \dots + c_{n+1-m,j}((\mu_{n+1-m})_t - (\nu_{n+1-m})_x) = 0.$$

Since $(\mu_i)_t - (\nu_i)_x$ is proportional to the left hand side of (3.24), where λ is set to λ_i , we obtain $m + 2$ linear dependences between coefficients of the left hand side of (3.24). Note that μ_i (resp. ν_i) are given by the formula (2.14) for f (resp. for g), where $h_j(\lambda, u_j)$ are replaced by $\frac{1}{\lambda_i - u_j}$ \square .

Here we present a more explicit form of the functions f and g . Let $S(\lambda) = (\lambda - \lambda_1) \dots (\lambda - \lambda_{n+1-m})$, where λ_i are pairwise distinct constants and $R(\lambda) = (\lambda - u_1) \dots (\lambda - u_n)$. Functions f, g can be written as

$$\begin{aligned} f(u_1, \dots, u_{n+1-m}, v_1, \dots, v_m, \lambda) &= \\ \frac{S(\lambda) \sum_{1 \leq i_1 < \dots < i_{m+1} \leq n+1-m} \phi_{i_1 \dots i_{m+1}} (\lambda - \lambda_{i_1}) \dots (\lambda - \lambda_{i_{m+1}}) R(\lambda_{i_1})^{-1} \dots R(\lambda_{i_{m+1}})^{-1}}{R(\lambda) \sum_{1 \leq i_1 < \dots < i_{m+2} \leq n+1-m} \Delta_{i_1 \dots i_{m+2}} R(\lambda_{i_1})^{-1} \dots R(\lambda_{i_{m+2}})^{-1}} \\ g(u_1, \dots, u_{n+1-m}, v_1, \dots, v_m, \lambda) &= \\ \frac{S(\lambda) \sum_{1 \leq i_1 < \dots < i_{m+1} \leq n+1-m} \psi_{i_1 \dots i_{m+1}} (\lambda - \lambda_{i_1}) \dots (\lambda - \lambda_{i_{m+1}}) R(\lambda_{i_1})^{-1} \dots R(\lambda_{i_{m+1}})^{-1}}{R(\lambda) \sum_{1 \leq i_1 < \dots < i_{m+2} \leq n+1-m} \Delta_{i_1 \dots i_{m+2}} R(\lambda_{i_1})^{-1} \dots R(\lambda_{i_{m+2}})^{-1}} \end{aligned} \quad (3.31)$$

where

$$\begin{aligned} \phi_{i_1 \dots i_{m+1}} &= \prod_{1 \leq \alpha < \beta \leq m+1} (\lambda_{i_\alpha} - \lambda_{i_\beta}) \det \begin{pmatrix} a_{i_1} & \dots & a_{i_{m+1}} \\ c_{i_1,1} & \dots & c_{i_{m+1},1} \\ \dots & \dots & \dots \\ c_{i_1,m} & \dots & c_{i_{m+1},m} \end{pmatrix}, \\ \psi_{i_1 \dots i_{m+1}} &= \prod_{1 \leq \alpha < \beta \leq m+1} (\lambda_{i_\alpha} - \lambda_{i_\beta}) \det \begin{pmatrix} b_{i_1} & \dots & b_{i_{m+1}} \\ c_{i_1,1} & \dots & c_{i_{m+1},1} \\ \dots & \dots & \dots \\ c_{i_1,m} & \dots & c_{i_{m+1},m} \end{pmatrix}, \end{aligned}$$

$$\Delta_{i_1 \dots i_{m+2}} = \prod_{1 \leq \alpha < \beta \leq m+2} (\lambda_{i_\alpha} - \lambda_{i_\beta}) \det \begin{pmatrix} a_{i_1} & \dots & a_{i_{m+2}} \\ b_{i_1} & \dots & b_{i_{m+2}} \\ c_{i_1,1} & \dots & c_{i_{m+2},1} \\ \dots & \dots & \dots \\ c_{i_1,m} & \dots & c_{i_{m+2},m} \end{pmatrix},$$

and $a_i, b_i, \lambda_i, c_{ij}$ are constants. In particular, if $m = 0$, we have

$$f(u_1, \dots, u_{n+1}, \lambda) = \frac{S(\lambda) \sum_{i=1}^{n+1} (\lambda - \lambda_i) a_i R(\lambda_i)^{-1}}{R(\lambda) \sum_{1 \leq i < j \leq n+1} (\lambda_i - \lambda_j) (a_i b_j - a_j b_i) R(\lambda_i)^{-1} R(\lambda_j)^{-1}} \quad (3.32)$$

$$g(u_1, \dots, u_{n+1}, \lambda) = \frac{S(\lambda) \sum_{i=1}^{n+1} (\lambda - \lambda_i) b_i R(\lambda_i)^{-1}}{R(\lambda) \sum_{1 \leq i < j \leq n+1} (\lambda_i - \lambda_j) (a_i b_j - a_j b_i) R(\lambda_i)^{-1} R(\lambda_j)^{-1}}$$

Note that this form is useful for changing coordinates. For example, we can choose $R(\lambda) = v_1 + v_2 \lambda + \dots + v_n \lambda^{n-1} + \lambda^n$.

3.2 Degenerations

The construction of Proposition 1 admits a limit when some of λ_i coincide. All formulas are valid in this case except (3.25). The formula (3.25) should be rewritten as

$$q_{n-m-3+j,i}(u) = \frac{\bar{c}_{1,j} + \bar{c}_{2,j}u + \dots + \bar{c}_{n+1-m,j}u^{n-m}}{(u - \lambda_1) \dots (u - \lambda_{n+1-m})}, \quad (3.33)$$

where $\bar{c}_{i,j}$ are constants. In (3.33) some of λ_i may coincide. An analog of (3.25) can be obtained from (3.33) by the partial fraction expansion of rational functions in u . Note that if $\lambda_i = \lambda_{i+1} = \dots = \lambda_{i+k}$, then the corresponding densities of conservation laws (3.21) read

$$\mu_{i,j} = \frac{\phi_1}{(\lambda_i - u_1)^j} + \dots + \frac{\phi_n}{(\lambda_i - u_n)^j}, \quad \nu_{i,j} = \frac{\psi_1}{(\lambda_i - u_1)^j} + \dots + \frac{\psi_n}{(\lambda_i - u_n)^j},$$

where $j = 1, \dots, k+1$.

It is clear from (1.5) that functions f, g of the form (3.23) with degrees of polynomials P, Q, R, S equal $m+1, m+1, n, n+1-m$ respectively transform as

$$f \rightarrow \bar{f} = \frac{(c\lambda + d)^2 S(\bar{\lambda}) P(\bar{\lambda})}{R(\bar{\lambda})}, \quad g \rightarrow \bar{g} = \frac{(c\lambda + d)^2 S(\bar{\lambda}) Q(\bar{\lambda})}{R(\bar{\lambda})}$$

under transformations (3.19). In particular, one of the values of λ_i can be sent to infinity. If $\lambda_i = \infty$ and this root has multiplicity k , say $\lambda_{n+1-m} = \dots = \lambda_{n+2-m-k} = \infty$, then $S(\lambda) = (\lambda - \lambda_1) \dots (\lambda - \lambda_{n+1-m-k})$ and (3.33) takes the form

$$q_{n-m-3+j,i}(u) = \frac{\bar{c}_{1,j} + \bar{c}_{2,j}u + \dots + \bar{c}_{n+1-m,j}u^{n-m}}{(u - \lambda_1) \dots (u - \lambda_{n+1-m-k})}.$$

Consider the case when all $\lambda_1 = \dots = \lambda_{n+1-m} = \infty$. We have $S(\lambda) = 1$ and

$$q_{n-m-3+j,i}(u) = \bar{c}_{1,j} + \bar{c}_{2,j}u + \dots + \bar{c}_{n+1-m,j}u^{n-m}.$$

Since we can replace the equations in system (2.17) by any of their linear combinations, in particular we can subtract linear combinations of equations (3.27) from other equations, we can assume without loss of generality that

$$q_{n-m-3+j,i}(u) = \bar{c}_{1,j}u^{n-m-2} + \bar{c}_{2,j}u^{n-m-1} + \bar{c}_{3,j}u^{n-m}.$$

Therefore we can only have $m = 0$ or $m = 1$. If $m = 0$ and our constants $c_{i,j}$ are chosen in such a way that

$$q_{n-2,i}(u) = u^{n-2}, \quad q_{n-1,i}(u) = u^{n-1},$$

then our system takes the form

$$u_{i,t} = \sum_{j \neq i} u_j u_{i,x} + \prod_{j \neq i} (u_i - u_j)^{-1}, \quad i = 1, \dots, n.$$

This system has appeared in [8], where a different problem was studied. If $n = 2$, then this system coincides with (1.3).

3.3 General scheme

The main idea of our construction from Proposition 1 can be described as follows. Let $L(\lambda)$ be the difference between the left and right hand sides of (3.18). If f, g are given by (3.23), then L is a rational function in λ and its numerator is the left hand side of (3.24). If the degrees of the polynomials P, Q, R, S are $m+1, m+1, n, n+1-m$ respectively, then the degree of the numerator of L equals $n+m+1$ and therefore the identity $L = 0$ is equivalent to a system of $n+m+2$ non-homogeneous hydrodynamic type equations. To provide $m+2$ linear relations between them we impose constraints of the form

$$c_{1,j}L(\lambda_1) + \dots + c_{n+1-m,j}L(\lambda_{n+1-m}) = 0, \quad j = 1, \dots, m+2,$$

where the $c_{i,j}$ are constants. Since $L(\lambda_i) = f(\lambda_i)_t - g(\lambda_i)_x$, these constraints follow from

$$c_{1,j}f(\lambda_1) + \dots + c_{n+1-m,j}f(\lambda_{n+1-m}) = a_j, \quad c_{1,j}g(\lambda_1) + \dots + c_{n+1-m,j}g(\lambda_{n+1-m}) = b_j, \quad (3.34)$$

where $j = 1, \dots, m+2$ and a_j, b_j are arbitrary constants.² The coefficients of P and Q are uniquely expressed from (3.34) in term of coefficients of R .

It is possible to generalize the construction described in Section 3.1 in the following way. We still assume that f and g are given by (3.23), where the degrees of polynomials P, Q, R, S

²Without loss of generality we can set $a_j = \delta_{m+1,j}$, $b_j = \delta_{m,j}$.

are $m+1$, $m+1$, n , $n+1-m$ and $0 \leq m \leq n-1$. In the case $m < n-1$ we use representation (3.26). If $m = n-1$, then

$$f = S(\lambda) \left(\phi_0 + \frac{\phi_1}{\lambda - u_1} + \dots + \frac{\phi_n}{\lambda - u_n} \right), \quad g = S(\lambda) \left(\psi_0 + \frac{\psi_1}{\lambda - u_1} + \dots + \frac{\psi_n}{\lambda - u_n} \right). \quad (3.35)$$

In our generalization we suppose that (3.34) is valid for $j = 1, \dots, m+2-k$. We look for the remaining k linear constraints in the form

$$\begin{aligned} d_{1,j,i} f(\lambda_i) + d_{2,j,i} f'(\lambda_i) + \dots + d_{l,j,i} \frac{f^{(l-1)}(\lambda_i)}{(l-1)!} &= 0, \\ d_{1,j,i} g(\lambda_i) + d_{2,j,i} g'(\lambda_i) + \dots + d_{l,j,i} \frac{g^{(l-1)}(\lambda_i)}{(l-1)!} &= 0, \end{aligned} \quad (3.36)$$

where $j = 1, \dots, k_i$, $i = 1, \dots, n+1-m$ and $k_1 + \dots + k_{n+1-m} = k$. The coefficients $d_{s,j,i}$ have to be found from the identity

$$d_{1,j,i} L(\lambda_i) + d_{2,j,i} L'(\lambda_i) + \dots + d_{l,j,i} \frac{L^{(l-1)}(\lambda_i)}{(l-1)!} = 0. \quad (3.37)$$

Let

$$\begin{aligned} f &= (\lambda - \lambda_i) f_0 + (\lambda - \lambda_i)^2 f_1 + \dots, & g &= (\lambda - \lambda_i) g_0 + (\lambda - \lambda_i)^2 g_1 + \dots, \\ L &= L_0 + (\lambda - \lambda_i) L_1 + (\lambda - \lambda_i)^2 L_2 + \dots. \end{aligned}$$

We omit the index i in the coefficients of these Taylor expansions for simplicity. Substituting the Taylor expansions into (3.18), we obtain

$$\begin{aligned} L_0 &= f_{0,t} - g_{0,x}, \\ L_i &= f_{i,t} - g_{i,x} + \sum_{0 \leq k < \frac{i}{2}} (i-2k)(f_k g_{i-k} - f_{i-k} g_k), \quad i > 0. \end{aligned}$$

Substituting these expressions into (3.37), we see that the terms $f_{j,t}$, $g_{j,x}$ cancel out by virtue of (3.36). The remaining parts of the expressions are bilinear in f_j , g_j . The vanishing of such parts leads to constraints for the coefficients $d_{s,j,i}$. Given a set of coefficients $d_{s,j,i}$ satisfying these constraints, we substitute expressions (3.26) for f , g into (3.36) and obtain a set of $k = k_1 + \dots + k_{n+1-m}$ linear equations for ϕ_i , ψ_i . We combine this system of linear equations with (3.27) and (3.30), where μ_i , ν_i are given by (3.29) and $j = 1, \dots, m+2-k$. The whole set of linear relations guarantees that the identity $L = 0$ is equivalent to a system of the form (1.2).

We do not describe here all admissible sets of coefficients $d_{s,j,i}$ in (3.36) but just discuss two admissible cases.

Case 1. Suppose that (3.36) takes a form $f_s = d_{s,i} f_0$, $g_s = d_{s,i} g_0$, $s = 1, \dots, k_i$. Then we have also $L_s = d_{s,i} L_0$, $s = 1, \dots, k_i$.

Let $m < n - 1$. In this case f, g are given by (3.26), where $\phi_1, \dots, \phi_n, \psi_1, \dots, \psi_n$ are defined as the solution of the linear system combined by the following three parts:

Part 1. The system (3.27) with $j = 0, \dots, n - m - 3$. This system is empty if $m = n - 2$.

Part 2. The system (3.30), where μ_i, ν_i are given by (3.29) and $j = 1, \dots, m + 2 - k$.

Part 3. The system

$$\begin{aligned} \left(\frac{1}{(\lambda_i - u_1)^{j+1}} - \frac{p_{j,i}}{\lambda_i - u_1} \right) \phi_1 + \dots + \left(\frac{1}{(\lambda_i - u_n)^{j+1}} - \frac{p_{j,i}}{\lambda_i - u_n} \right) \phi_n &= 0, \\ \left(\frac{1}{(\lambda_i - u_1)^{j+1}} - \frac{p_{j,i}}{\lambda_i - u_1} \right) \psi_1 + \dots + \left(\frac{1}{(\lambda_i - u_n)^{j+1}} - \frac{p_{j,i}}{\lambda_i - u_n} \right) \psi_n &= 0, \end{aligned} \quad (3.38)$$

where $j = 1, \dots, k_i, i = 1, \dots, n + 1 - m$ and $k_1 + \dots + k_{n+1-m} = k$.

If $m = n - 1$ then we have to take $k = n - 1$. The functions f, g are given by (3.35), where $\phi_0, \dots, \phi_n, \psi_0, \dots, \psi_n$ is the solution of the linear system combined by the following two parts:

Part 1. The system (3.30), where μ_i, ν_i are given by

$$\mu_i = \phi_0 + \frac{\phi_1}{\lambda_i - u_1} + \dots + \frac{\phi_n}{\lambda_i - u_n}, \quad \nu_i = \psi_0 + \frac{\psi_1}{\lambda_i - u_1} + \dots + \frac{\psi_n}{\lambda_i - u_n}$$

and $j = 1, 2$.

Part 2. The system

$$\begin{aligned} -p_{j,i}\phi_0 + \left(\frac{1}{(\lambda_i - u_1)^{j+1}} - \frac{p_{j,i}}{\lambda_i - u_1} \right) \phi_1 + \dots + \left(\frac{1}{(\lambda_i - u_n)^{j+1}} - \frac{p_{j,i}}{\lambda_i - u_n} \right) \phi_n &= 0, \\ -p_{j,i}\psi_0 + \left(\frac{1}{(\lambda_i - u_1)^{j+1}} - \frac{p_{j,i}}{\lambda_i - u_1} \right) \psi_1 + \dots + \left(\frac{1}{(\lambda_i - u_n)^{j+1}} - \frac{p_{j,i}}{\lambda_i - u_n} \right) \psi_n &= 0, \end{aligned} \quad (3.39)$$

where $j = 1, \dots, k_i, i = 1, 2$ and $k_1 + k_2 = n - 1$. \square

Case 2. Suppose that (3.36) takes the form

$$f_s = a_{s,i}f_0 + b_{s,i}f_1, \quad g_s = a_{s,i}g_0 + b_{s,i}g_1, \quad s = 2, \dots, k_i, \quad (3.40)$$

where the coefficients $a_{i,j}$ and $b_{p,q}$ satisfy the following equations

$$(s - 2)a_{s-1,i} + b_{s,i} + \sum_{2 \leq k < \frac{s}{2}} (s - 2k)(b_{k,i}a_{s-k,i} - b_{s-k,i}a_{k,i}) = 0, \quad s = 2, 3, \dots$$

In particular, $b_{2,i} = 0$. Then we have also $L_s = a_{s,i}L_0 + b_{s,i}L_1, s = 2, \dots, k_i$. The explicit form of a linear system for $\phi_1, \dots, \phi_n, \psi_1, \dots, \psi_n$, where $m < n - 1$ (resp. for $\phi_0, \dots, \phi_n, \psi_0, \dots, \psi_n$, where $m = n - 1$) can be obtained straightforwardly by substitution of (3.26) (resp. (3.35)) into (3.40) and combining these equations with (3.27) and (3.30). \square

Remark 3. For any admissible coefficients $d_{s,j,i}$, if λ_i are distinct, then we must have

$$2m + 1 - n \leq k \leq m, \quad k_1 + \dots + k_{n+1-m} = k. \quad (3.41)$$

Indeed, we must have at least two equations of the form (3.30) for ϕ_i (resp. for ψ_i), so $k \leq m$. On the other hand, the rank of the matrix $(c_{j,i})$ should be equal to $m + 2 - k$. Therefore, $m + 2 - k \leq n + 1 - m$ or $2m + 1 \leq n + k$. In particular, if $m = n - 1$, then $k = n - 1$. \square

Consider the case $n = 2$. It is easy to see that (3.41) admits only two solutions:

1. $m = 0$, $k = k_1 = k_2 = k_3 = 0$. This is the case of Proposition 1. Explicit formulas are given by (4.46).

2. $m = 1$, $k = 1$. Without loss of generality we can set $k_1 = 1$, $k_2 = 0$. There are only two possibilities for coefficients $d_{s,j,i}$ and both are described by the cases 1 and 2 above. In case 1 we have $f_1 = af_0$, $g_1 = ag_0$ where a is an arbitrary constant and f_i , g_i are coefficients of the Taylor expansion of f , g respectively at $\lambda = \lambda_1$. Explicit formulas are given by (4.49). In case 2 we have $f_2 = af_0$, $g_2 = ag_0$, where a is an arbitrary constant and f_i , g_i are coefficients of the Taylor expansion of f , g respectively at $\lambda = \lambda_1$. Explicit formulas are given by (4.47).

4 Classification in the case $n = 2$

Consider the case $n = 2$. In this case we do not assume that f and g are rational functions in λ . Denote $u_1 = u$, $u_2 = v$. Expanding the functional equation (2.16) a neighborhood of the diagonal $v = u$, we obtain $q_{1,1}(u) = q_{1,2}(u)$ and $h_1(\lambda, u) = h_2(\lambda, u)$. Let

$$q_{1,1}(u) = q_{1,2}(u) = \frac{1}{a(u)}, \quad h_1(\lambda, u) = h_2(\lambda, u) = \frac{h(\lambda, u)}{a(u)}.$$

Then system (1.2) takes the form

$$u_t = a(v)u_x + h(u, v), \quad v_t = a(u)v_x + h(v, u) \quad (4.42)$$

and the corresponding Lax pair (1.5) is defined by

$$f(u, v, \lambda) = \frac{h(\lambda, u) - h(\lambda, v)}{a(u) - a(v)}, \quad g(u, v, \lambda) = \frac{a(v)h(\lambda, u) - a(u)h(\lambda, v)}{a(u) - a(v)}. \quad (4.43)$$

Here h is a function with a simple pole on the diagonal. Formula (2.16) yields the following functional equation for this function:

$$\begin{aligned} h(\lambda, v)h(\lambda, u)_\lambda - h(\lambda, u)h(\lambda, v)_\lambda + h(u, v)h(\lambda, u)_u - h(v, u)h(\lambda, v)_v - \\ \frac{h(u, v)a'(u) - h(v, u)a'(v)}{a(u) - a(v)}(h(\lambda, u) - h(\lambda, v)) = 0. \end{aligned} \quad (4.44)$$

Remark 4. Suppose (4.44) holds and h has a simple pole on the diagonal. Then the following Gibbons-Tsarev type system with two fields u, v [6, 3, 4] is compatible:

$$\partial_i p_j = h(p_j, p_i) \partial_i u, \quad \partial_i v = a(p_i) \partial_i u, \quad \partial_i \partial_j u = \frac{h(p_i, p_j) a'(p_i) - h(p_j, p_i) a'(p_j)}{a(p_i) - a(p_j)} \partial_i u \partial_j u. \quad (4.45)$$

Here $i \neq j = 1, \dots, N$; p_1, \dots, p_N, u are functions in r_1, \dots, r_N , $\partial_i = \frac{\partial}{\partial r_i}$ and N is arbitrary. Note that there exist Gibbons-Tsarev type systems that have a slightly different structure. \square

Proposition 2. For each Gibbons-Tsarev type system of the form (4.45) one can construct a non-homogeneous system (2.9) with $n = 2$ possessing a Lax representation.

Proof. Set $N = 2$ in (4.45) and consider p_1, p_2 as functions of u, v . We have

$$\partial_i p_j = (p_j)_u \partial_i u + (p_j)_v \partial_i v = (p_j)_u \partial_i u + (p_j)_v a(p_i) \partial_i u = h(p_j, p_i) \partial_i u$$

or $(p_j)_u + (p_j)_v a(p_i) = h(p_j, p_i)$, where $i \neq j = 1, 2$. Moreover, it is known [4] that each Gibbons-Tsarev type system admits so-called dispersionless Lax operator. It is a function $L(\lambda, r_1, \dots, r_N)$ defined (up to transformations of the form $L \rightarrow q(L)$) by the following system

$$\partial_i L = h(\lambda, p_i) L_\lambda \partial_i u, \quad i = 1, \dots, N.$$

Note that this system is compatible by virtue of (4.45). Taking $N = 2$, we may consider L as a function of u, v, λ . As above we obtain $L_u + L_v a(p_i) = h(\lambda, p_i) L_\lambda$, $i = 1, 2$ or $L_u = f(p_1, p_2, \lambda) L_\lambda$, $L_v = g(p_1, p_2, \lambda) L_\lambda$, where

$$f = \frac{a(p_2)h(\lambda, p_1) - a(p_1)h(\lambda, p_2)}{a(p_2) - a(p_1)}, \quad g = \frac{h(\lambda, p_1) - h(\lambda, p_2)}{a(p_1) - a(p_2)}. \quad \square$$

Remark 5. In [8] several systems of the form (4.42) possessing a conservation law with density and flux that depend on u, v, u_x, v_x were found. The existence of such a conservation law was proposed as an indication of complete integrability. Most of the systems from [8] do not satisfy our functional equation (4.44). \square

Let us present several solutions of the functional equation (4.44).

Example 1. The functions

$$h(u, v) = \frac{S_3(u) S_3(v)}{W_2(v)(u - v)}, \quad a(u) = \frac{P_2(u)}{W_2(u)}. \quad (4.46)$$

where S_3, P_2 and W_2 are arbitrary polynomials of degree 3, 2 and 2 correspondingly, satisfy (4.44). For equation (1.3) we have $S_3(u) = 1$, $W_2(u) = -1$, $P_2(u) = -u$. \square

Example 2. Let

$$h(u, v) = \frac{Q(u) Q(v) P(u) (P(v)^2 P(u) + (u - v) Q(v))}{R(v)(u - v)}, \quad a(u) = \frac{S(u)}{R(u)}, \quad (4.47)$$

where $P(u) = p_1u + p_0$, $Q(u) = q_1u + q_0$ are arbitrary polynomials and the coefficients of polynomials $S(u) = s_2u^2 + s_1u + s_0$ and $R(u) = r_2u^2 + r_1u + r_0$ satisfy the same linear equation

$$2p_0q_0x_2 - (p_0q_1 + p_1q_0)x_1 + 2p_1q_1x_0 = 0. \quad (4.48)$$

Then the functions h, a satisfy (4.44). Notice that (4.48) means that the double ratio (pq, z_1z_2) equals -1 , where p, q, z_1, z_2 are roots of the polynomials $P(u), Q(u)$ and $x_2u^2 + x_1u + x_0$ respectively. \square

Example 3. The functions

$$h(u, v) = \frac{Q(u)Q(v)R(u)\left(R(v)R(u) + k(u-v)\right)}{T(v)(u-v)}, \quad a(u) = \frac{S(u)}{T(u)}, \quad (4.49)$$

where $Q(u) = q_1u + q_0$, $S(u) = s_1u + s_0$, $R(u) = r_1u + r_0$, $T(u) = t_1u + t_0$ are arbitrary polynomials, satisfy (4.44). \square

It is easy to verify that the classes of solutions described in Examples 1-3 are invariant with respect to the transformations

$$u \rightarrow \frac{k_1u + k_2}{k_3u + k_4}, \quad v \rightarrow \frac{k_1v + k_2}{k_3v + k_4} \quad (4.50)$$

and

$$x \rightarrow r_1x + r_2t, \quad t \rightarrow r_3x + r_4t, \quad (4.51)$$

where k_i and r_i are arbitrary constants.

Remark 6. Using transformations (4.50), (4.51), we can bring some of polynomials from Examples 1-3 to a canonical form. For instance, in the generic case of Example 1 one can bring polynomial S_3 to $S_3(u) = u(u-1)$. \square

Theorem 1. Any solution of (4.44) that has a simple pole on the diagonal is given by (4.46), (4.47) or (4.49) up to transformations (4.53).

Proof. Let

$$h(z, x) = \frac{a_{-1}(x)}{z-x} + a_0(x) + a_1(x)(z-x) + \dots \quad (4.52)$$

Relations (4.42), (4.43), (4.44) admit arbitrary transformations of the form

$$u \rightarrow \phi(u), \quad v \rightarrow \phi(v). \quad (4.53)$$

Normalizing $a(u)$ by u with the help of (4.53) and expanding our functional equation at $\lambda = u$, we obtain

$$h(u, v)_u = \frac{1}{2a_{-1}(u)(u-v)} \left(a_{-1}(u)h(v, u) + ((u-v)a'_{-1}(u) - a_{-1}(u))h(u, v) \right). \quad (4.54)$$

For the function

$$q(u, v) = \frac{h(u, v)}{\sqrt{a_{-1}(u)a_{-1}(v)}} \quad (4.55)$$

equation (4.54) takes the form

$$q(u, v)_u = \frac{q(v, u) - q(u, v)}{2(u - v)}. \quad (4.56)$$

From (4.56) it follows that $q(u, v)_u = q(v, u)_v$. Differentiating (4.56) by v and eliminating $q(v, u)$ and $q(v, u)_v$, we arrive at the following Euler-Darboux equation

$$q_{uv} = \frac{3}{2} \frac{q_u}{u - v} - \frac{1}{2} \frac{q_v}{u - v} \quad (4.57)$$

for $q(u, v)$.

We are interested in solutions of (4.57) of the form $q(u, v) = \frac{1}{u - v} + G(u, v)$, where $G(u, v)$ is holomorphic on the diagonal $u = v$. It is easy to verify that for any such a solution the function G is of the form

$$G(u, v) = \sum_0^\infty \frac{s(u + v)^{(2k)}}{(k!)^2 2^{2k}} (u - v)^{2k} - \sum_1^\infty \frac{s(u + v)^{(2k-1)}}{k!(k-1)! 2^{2k-1}} (u - v)^{2k-1}$$

for some function $s(x)$. The functions $a_{-1}(x), a_0(x)$ from (4.44) and $s(x)$ are related as follows

$$a_0 = \frac{a'_{-1}}{2} + s a_{-1}. \quad (4.58)$$

We will show that for any solution (4.52) of (4.44) the corresponding function s is rational (in contrast with a_{-1}, a_0). For any rational $s(x)$ the corresponding function G can be easily reconstructed in closed form (cf. [9]). In particular, if $s(x) = 1/(x - k)$ then $G(u, v) = T(u, v, k)$, where

$$T(u, v, k) = \frac{2}{(\sqrt{u - k} + \sqrt{v - k})\sqrt{v - k}}. \quad (4.59)$$

The multiple pole $s(x) = 1/(x - k)^2$ corresponds to $\frac{\partial T(u, v, k)}{\partial k}$ and so on.

Substituting (4.52), (4.58) into (4.44) and expanding in a small neighborhood of $u = v = z$, we find that all coefficients a_i , with $i > 0$ are uniquely determined through a_{-1} and s . These two functions satisfy an overdetermined system of ODEs. Eliminating a_{-1} , we arrive at another system for s only. This system contains two ODEs of fifth and fourth orders. Differentiating the fourth order ODE and eliminating the fifth derivative by fifth order equation, we get several more fourth order ODEs. The final system is so complicated that usual algorithms of computer algebra such as differential Groebner basis technique certainly do not work. Fortunately, the system admits a group of point symmetries.

Although we fix $a(u) = u$, the equation (4.44) still admits symmetries. Namely, after any transformation of the form (4.51) the function $a(u)$ becomes fractional-linear and we can bring it back to u by an appropriate transformation (4.50). The existence of this $GL(2)$ -action implies the symmetry group

$$s(u) \rightarrow \frac{k_3}{2(k_3u + k_4)} + \frac{(k_1k_4 - k_2k_3) s\left(\frac{k_1u + k_2}{k_3u + k_4}\right)}{(k_3u + k_4)^2}$$

of the ODE system for $s(u)$. Taking the simplest differential invariants

$$x = \frac{s'' + 12ss' + 16s^3}{(s' + 2s^2)^{3/2}}, \quad y = \frac{s''' + 24ss'' + 144s^2s' + 144s^4}{(s' + 2s^2)^2}$$

of this action for new dependent and independent variables, we can reduce the order of equations by 3. The equations of fourth order turn into very complicated first order equations for the function $y = G(x)$. Considering these equations as polynomials in x, G and G' and eliminating G' by using the resultant several times, we find that the whole system is equivalent to the following single algebraic equation

$$\begin{aligned} &G^6 + 288G^5 - 216(5x^2 - 124)G^4 + 54(13x^4 - 944x^2 + 16768)G^3 \\ &- 3888(27x^4 + 40x^2 - 3632)G^2 + 5832(63x^6 - 1124x^4 + 1856x^2 + 17920)G \\ &- 729(343x^8 - 9376x^6 + 69120x^4 - 118784x^2 - 409600) = 0. \end{aligned}$$

The general solution of this third order ODE for $s(u)$ is given by

$$s(u) = \frac{1}{4(u - k_1)} + \frac{1}{4(u - k_2)} + \frac{k_3}{(u - k_1)^2}, \quad (4.60)$$

where k_i are arbitrary constants. Besides (4.60) there is a special solution of the form

$$s(u) = \frac{1}{4(u - k_1)} + k_2. \quad (4.61)$$

The next step is to find the function a_{-1} from the initial ODE system for $s(u)$ and $a_{-1}(u)$. It turns out that if s is given by (4.60) with $k_3 = 0$ then this system is equivalent to a single non-linear ODE of fourth order for a_{-1} . It can be linearized by the substitution $a_{-1} = (u - k_1)(u - k_2)P(u)^2$. Solving the linear ODE for P , we find

$$a_{-1}(u) = (u - k_1)(u - k_2) \left(P_1(u) \sqrt{u - k_1} + P_2(u) \sqrt{u - k_2} \right)^2,$$

where P_i are arbitrary first degree polynomials. The function $g(u, v)$ is reconstructed from s as follows

$$g(u, v) = \frac{1}{u - v} + \frac{1}{4}T(u, v, k_1) + \frac{1}{4}T(u, v, k_2),$$

where $T(u, v, k)$ is given by (4.59). The function $h(u, v)$ is defined by (4.55). The radicals $\sqrt{u - k_1}$ and $\sqrt{u - k_2}$ can be removed by an appropriate transformation of the form (4.53) and as result we get the solution described in Example 1.

Example 2 corresponds to (4.60) with $k_3 \neq 0$. In this case we have

$$g(u, v) = \frac{1}{u - v} + \frac{1}{4}T(u, v, k_1) + \frac{1}{4}T(u, v, k_2) + k_3 \frac{\partial T(u, v, k_1)}{\partial k_1}$$

and $a_{-1} = \alpha(u - k_1)^3(u - k_2)^2$, where α is arbitrary constant.

The degeneration $k_2 = k_1$, $k_3 \neq 0$ of (4.60) gives rise to Example 3. In this case $a_1 = (u - k_1)^3 P_1^2$, where P_1 is an arbitrary first degree polynomial.

Solutions of (4.44) corresponding to other degenerations can be written in one of the forms (4.46), (4.47) or (4.49) with special polynomials there. In particular, the degeneration $k_2 = k_1$, $k_3 = 0$ corresponds to (4.46), where the polynomials P_2 and Q_2 have a common root. \square

5 Examples in the case $n = 3$

In the case $N = 3$ integrable systems have the form

$$\begin{aligned} u_t &= a_1(v, w)u_x + b_1(u, v, w), \\ v_t &= a_2(u, w)v_x + b_2(u, v, w), \\ w_t &= a_3(u, v)w_x + b_3(u, v, w), \end{aligned} \tag{5.62}$$

where

$$\begin{aligned} a_1(u, v) &= a_2(u, v) = a_3(u, v) = \frac{B(u) - B(v)}{A(u) - A(v)}, \\ b_1(u, v, w) &= \frac{X(v, u) - X(w, u)}{A(v) - A(w)}, \quad b_2(u, v, w) = \frac{X(w, v) - X(u, v)}{A(w) - A(u)}, \\ b_3(u, v, w) &= \frac{X(u, w) - X(v, w)}{A(u) - A(v)}. \end{aligned}$$

The Lax representation of the form (1.5) is given by

$$\begin{aligned} f(u, v, w, \lambda) &= \frac{X(u, \lambda)(A(v) - A(w)) + X(v, \lambda)(A(w) - A(u)) + X(w, \lambda)(A(u) - A(v))}{B(u)(A(v) - A(w)) + B(v)(A(w) - A(u)) + B(w)(A(u) - A(v))}, \\ g(u, v, w, \lambda) &= \frac{X(u, \lambda)(B(v) - B(w)) + X(v, \lambda)(B(w) - B(u)) + X(w, \lambda)(B(u) - B(v))}{B(u)(A(v) - A(w)) + B(v)(A(w) - A(u)) + B(w)(A(u) - A(v))}. \end{aligned}$$

In the above formulas the functions $X(u, \lambda)$, $A(u)$, $B(u)$ satisfy a complicated functional equation, which follows from (2.16).

Remark 7. Taking into account transformations (4.51), we see that the following group of affine transformations is admissible:

$$A \rightarrow c_1 A + c_2 B + c_3, \quad B \rightarrow c_4 A + c_5 B + c_6,$$

5.1 A class of solutions

Let us describe explicitly all integrable equations with the function X of the form

$$X(u, \lambda) = \frac{R(u)S(\lambda)}{\lambda - u} \quad (5.63)$$

for some functions R, S . In this section we do not assume that the functions R and S are polynomials. Note that the transformation of X under (4.50) is given by

$$R(u) \rightarrow (k_3 u + k_4) R\left(\frac{k_1 u + k_2}{k_3 u + k_4}\right), \quad S(\lambda) \rightarrow (k_3 \lambda + k_4)^3 S\left(\frac{k_1 \lambda + k_2}{k_3 \lambda + k_4}\right). \quad (5.64)$$

It can be straightforwardly verified that all examples found below belong to the class of equations described in Section 3.

From (2.16) it follows that the functions R, S satisfy several ODEs linear in S . The simplest of them are

$$\begin{aligned} 6RR''S^{(5)} + 5(3R'R'' - RR''')S^{(4)} &= 0, \\ R^2S^{(6)} + 12RR'S^{(5)} + 15(2R'^2 - RR'')S^{(4)} &= 0, \\ RR''S^{(6)} - 8RR'''S^{(5)} + 5(RR'''' - 2R'R''' - 3R''^2)S^{(4)} &= 0, \end{aligned} \quad (5.65)$$

and

$$\begin{aligned} R^4 S^{(5)} + 5R^3 R' S^{(4)} - 20R^3 R'' S^{(3)} + 10R^2 (3R'R'' - RR''') S'' - \\ 5R(R^2 R^{(4)} - 12RR''^2 + 12R'^2 R'') S' - (R^3 R^{(5)} - 30R^2 R'' R''' + 90RR'R''^2 - 60R'^3 R'') S. \end{aligned} \quad (5.66)$$

There are two different possibilities: **Case A:** $S'''' = 0$ and **Case B:** $S'''' \neq 0$. In Case B the determinant of the system of linear equations (5.65) for $S^{(4)}, S^{(5)}, S^{(6)}$ should be zero. This leads to a fourth order ODE for R , whose solution is $R = W_2/W_1$, where W_2 and W_1 are arbitrary polynomials of degree 2 and 1, correspondingly. Using a transformation of the form (5.64), we bring W_1 to 1.

In Case A besides (5.66) and $S^{(4)} = 0$ we use one more equation linear in S . This equation of order 3 in S and order 4 in R is rather complicated and we do not present it here. Differentiating these equations, we get several more linear equations. Eliminating S and its derivatives from the system thus obtained, we arrive at an overdetermined system of non-linear ODEs for R . Investigating the compatibility of latter system, we find all possible functions R . Given R the corresponding cubic polynomial S can be easily found from (5.66).

One of possible pairs R, S in Case A is given by **Case A-1:**

$$R = \frac{S_3}{W}, \quad S = S_3,$$

where S_3 and W are arbitrary polynomials of degree 3 and 2 (cf. Example 1).

It is possible to show that otherwise we have **Case A-2**: $R = 1/W$ up to transformations of the form (5.64). Notice that the polynomial W can be reduced by a linear transformation (5.64) to one of the following canonical forms: $W(x) = x(x-1)$, $W(x) = x^2$, $W(x) = x$, or $W(x) = 1$. Using (5.66), we find the following solutions:

$$\begin{aligned} R(x) &= \frac{1}{x(x-1)}, & S(x) &= c_1 + c_2x(x-1), \\ R(x) &= \frac{1}{x^2}, & S(x) &= c_1 + c_2x^2, \\ R(x) &= \frac{1}{x}, & S(x) &= c_1 + c_2x + c_3x^2. \end{aligned}$$

For $W = 1$ see Case B.

It is easy to verify that in Case B the only solution of (5.65), (5.66) is $R = 1, S = S_4$, where S_4 is an arbitrary fourth degree polynomial.

Now we should find the functions A and B for all above cases. It can be verified that A and B satisfy the same fifth order linear equation whose coefficients are differential polynomials in R, S . Hence all possible functions $A(u)$ form a vector space V of dimension ≤ 5 . According to Remark 7 all possible functions B belong to V and $\dim V$ has to be not less than 3.

It turns out that in Case B we have

$$R = 1, \quad S = S_4, \quad A = \frac{A_4}{S_4}, \quad B = \frac{B_4}{S_4},$$

where S_4, A_4, B_4 are arbitrary polynomials of fourth degree. In this case $\dim V = 5$.

In Case A-2 we obtain that $\dim V = 4$. If $\deg W = 2$ then a basis of V is given by

$$A_1 = 1, \quad A_2 = \frac{1}{W}, \quad A_3 = \frac{1}{SW}, \quad A_4 = \frac{u}{SW}.$$

In the case $\deg W < 2$ a basis is $\frac{u^i}{SW}$, $i = 0, 1, 2, 3$.

For Case A-1 (if the pair R, S does not belong to Cases B and A-2), then the vector space V is three-dimensional with a basis $\frac{u^i}{W}$, $i = 0, 1, 2$ (cf. Example 1).

The following example has a structure more complicated than (5.63):

Example 4.

$$X(u, z) = \frac{(u-a)^2(z-a)(z-b) \left((a+b)uz - 2b^2z - 2abu + b^2(a+b) \right)}{(u-z)u},$$

a basis of V is $u, 1, u^{-1}$.

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References

- [1] *B.A. Dubrovin, S.P. Novikov*, Hydrodynamics of weakly deformed soliton lattices: differential geometry and Hamiltonian theory, Russian Math. Surveys, **44** No. 6 (1989) 35-124.
- [2] *S.P. Tsarev*, The geometry of Hamiltonian systems of hydrodynamic type. The generalized hodograph method, Math. USSR Izvestiya, **37** No. 2 (1991) 397-419. 1048-1068.
- [3] *E.V. Ferapontov, K.R. Khusnutdinova*, On integrability of (2+1)-dimensional quasilinear systems, Comm. Math. Phys. **248** (2004) 187-206, *E.V. Ferapontov, K.R. Khusnutdinova*, The characterization of 2-component (2+1)-dimensional integrable systems of hydrodynamic type, J. Phys. A: Math. Gen. **37**(8) (2004) 2949 - 2963.
- [4] *A.V. Odesskii, V.V. Sokolov*, Intgerable (2+1)-dimensional systems of hydrodynamic type, Theor. and Math. Phys., **162**, no.2, (2010) 549-586.
- [5] *A.V. Odesskii, V.V. Sokolov*, Classification of integrable hydrodynamic chains, Journal Phys. A: Math. Gen., **43**, (2010) 434027, 15 pp.
- [6] *J. Gibbons, S.P. Tsarev*, Reductions of Benney's equations, Phys. Lett. A, **211** (1996) 19-24.
- [7] *A.V. Mikhailov, V.V. Sokolov, A.B. Shabat*, The symmetry approach to classification of integrable equations, in What is Integrability? Editor: Zakharov V.E., Springer series in Nonlinear Dynamics, 1991, 115-184.
- [8] *E. Ferapontov, A.P. Fordy*, Nonhomogeneous systems of hydrodynamic type related to quadratic Hamiltonians with electromagnetic term, Physica D., **108**(1997) 350-364.
- [9] *V.G. Marikhin, V.V. Sokolov*, Separation of variables on a non-hyperelliptic curve, Reg. and Chaot. Dynamics., **10**, no.1, (2005) 59-70. ArXiv nlin. SI/0412065
- [10] *J. Gibbons, S.P. Tsarev*, Conformal maps and reductions of the Benney equations, Phys. Lett. A, **258** (1999) 263-270.
- [11] *K. Löwner*, Untersuchungen über schlichte Konforme Abbildungen des Einheitskreises, Math. Ann., **89**(1923) 103-121.
- [12] *T. Takebe, L.-P. Leo and A. Zabrodin*, Löwner equations and dispersionless hierarchies, J. Phys. A: Math. Gen., **39**(2006) 11479-11501.
- [13] *A. Kokotov, D. Korotkin*, A new hierarchy of integrable systems associated to Hurwitz spaces, Philosophical Transactions of The Royal Society A: Mathematical, Physical and Engineering Sciences, **366**, no.1867, (2008) 1055-1088.

- [14] *S.P. Burtsev, V.E. Zakharov, A.V. Mikhailov*, Inverse scattering method with variable spectral parameter, *Theor. and Math. Phys.*, **70**, no.**3**, (1987) 323–341.
- [15] *A. Odesskii, V. Sokolov*, On (2+1)-dimensional hydrodynamic-type systems possessing pseudopotential with movable singularities, *Func. Anal. Appl.*, **42**, no.**3**, (2008) 205–212..
- [16] *S. V. Manakov and P. M. Santini*, Inverse scattering problem for vector fields and the Cauchy problem for the heavenly equation, *Phys. Lett. A.*, **359**, no.**6**, (2006) 613–619.
- [17] *E. V. Ferapontov*, Integration of weakly nonlinear hydrodynamic systems in Riemann invariants, *Phys. Lett. A.*, **158** (1991) 112–118.
- [18] *V. Shramchenko*, Integrable systems related to elliptic branched coverings, *J. Phys. A: Math. and Gen.*, **36** (42) (2003), 10585–10605.
- [19] *T. Takebe, L.-P. Leo and A. Zabrodin*, Löwner equations and dispersionless hierarchies, *J. Phys. A: Math. Gen.*, **39**(2006) 11479–11501.